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Asymptotic Upper Bound of Density for Two-Particle Annihilating Exclusion

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We consider a stochastic process which presents an evolution of particles of two types on $\mathbb{Z}^d$ with annihilations between particles of opposite types. Initially, at each site of $\mathbb{Z}^d$, independently of the other sites, we put a particle with probability $2\rho \leqslant 1$ and assign to it one of two types with equal chances. Each particle, independently from the others, waits an exponential time with mean 1, chooses one of its neighboring sites on the lattice $\mathbb{Z}^d$ with equal probabilities, and jumps to the site chosen. If the site to which a particle attempts to move is occupied by another particle of the same type, the jump is suppressed; if it is occupied by a particle of the opposite type, then both are annihilated and disappear from the system. The considered process may serve as a model for the chemical reaction $A + B \to$ inert. The paper concerns an upper bound of $\rho(t)$, the density of particles in the system at time t. We prove that $\rho(t) < t^{-d/4}t^{\varepsilon}$ when $t > t(\varepsilon)$ for all $\varepsilon > 0$ in the dimensions $d \leqslant 4$ and asymptotically $\rho(t) < Ct^{-1}$ in the higher dimensions. In our proofs, we used the ideas and the technique developed by Bramson and Lebowitz and the tools which are customarily used to study a symmetric exclusion process.

KEY WORDS: Diffusion-dominated reaction; two-particle annihilating exclusion; asymptotic upper bound of the density.

1. INTRODUCTION AND SUMMARY

Consider a stochastic process in which particles of two types, say A type and B type, participate. The particles of the same type execute a symmetric simple exclusion process on $\mathbb{Z}^d$ (in what follows, this process will be called simply *an exclusion process*) which evolves independently of the motion of the particles of the opposite type unless a particle jumps to a site which is

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currently occupied by a particle of the opposite type. If the latter happens, both particles annihilate and disappear. We call this process *two-particle annihilating exclusion* or simply *annihilating exclusion.*

We construct an initial configuration by putting particles in $\mathbb{Z}^d$ according to the Bernoulli product measure with the density $2\rho \leqslant 1$. Then, for each particle that is present in the configuration constructed, we choose its type to be either A or B with equal probabilities and independently of the type of the other particles. This procedure determines the initial distribution of the considered process.

Let us define the density of A (or B) particles in the system at time t by

$$\begin{aligned}\rho(t) &= \Pr[\text{the site 0 is occupied by A-type particle at time } t]\\ &= \Pr[\text{the site 0 is occupied by B-type particle at time } t]\end{aligned}$$

Our concern is with an upper bound of $\rho(t)$ as $t \to \infty$.

Theorem 1.1. (i) For every $\varepsilon > 0$ there is $T(\varepsilon) < \infty$ such that for $t > T(\varepsilon)$,

$$\rho(t) \leqslant t^{-d/4} t^{\varepsilon} \qquad \text{when} \quad d \leqslant 4 \tag{1.1a}$$

(ii) There is a positive constant C^* which may depend on d, such that for sufficiently large t,

$$\rho(t) \leqslant C^* t^{-1} \qquad \text{when} \quad d > 4 \tag{1.1b}$$

In the proofs, we will substitute an exclusion process by a stirring process. The advantage of this substitution is that in the latter, the marginal motion of a particle (until its annihilation) coincides with a random walk. This fact made it possible to utilize for our needs the ideas and tools used by Bramson and Lebowitz[4] for deriving the asymptotic behavior of the density for two-particle annihilating random walks. The latter is a process in which particles of two types execute independent simple random walks in $\mathbb{Z}^d$ and, analogously to our process, an annihilation occurs when two particles of opposite types attempt to occupy the same site. For this process, it is proven in ref. 4 that

$$ct^{-d/4} \leqslant \rho_{\text{indep}}(t) \leqslant Ct^{-d/4} \qquad \text{for} \quad d \leqslant 4 \tag{1.2a}$$

$$ct^{-1} \leqslant \rho_{\text{indep}}(t) \leqslant Ct^{-1} \qquad \text{for} \quad d \geqslant 4 \tag{1.2b}$$

when t is sufficiently large, for appropriate absolute constants $c = c(d)$ and $C = C(d)$. (In fact, the initial distribution for the process considered in ref. 4

is different from that considered in our paper. However, it is easy to check that (1.2a) and (1.2b) are true if the particles in the annihilating random walks are distributed initially as described in the beginning of this section.)

It turns out that majority of tools and ideas which we borrowed from ref. 4 suited our process. When, however, they failed to work, the correlation inequality substituted favorably. (The latter inequality is also called the Liggett inequality. It is formulated in Lemma 4.12 of ref. 7, Chapter VIII.) The only exception was estimating from above the mathematical expectation of the quantity

$$\begin{aligned} |\#(\text{A particles}) - \#(\text{B particles}) \text{ at time } t \\ \text{in the cube in } \mathbb{R}^d \text{ of side const} \times \sqrt{t}| \end{aligned} \tag{1.3}$$

In the case of Bramson and Lebowitz, the estimate const $\times\, t^{-d/4}$ is derived rather easily (see Lemma 2.2 in ref. 4). In our case, we needed a method that estimates quantitively the number of particles of an exclusion process which are in an arbitrarily given finite region of $\mathbb{Z}^d$ at an arbitrarily given time $t > 0$, provided the initial configuration of this exclusion process is known. In such a situation, it is usual to compare the positions of particles that interact by the exclusion rules with the positions of particles that walk independently (given that the initial positions of the particles are the same for both processes). In our previous work,[2] we used the tools which had been developed in ref. 5 for the needs of this comparison (see also ref. 8, p. 192). The results we obtained were $\rho(t) \leqslant Ct^{-1/8}$ in $d = 1$ and $\leqslant Ct^{-1/4}$ in $d > 1$. In the present paper, we use a method which we learned from Andjel.[1] This method utilizes integration by parts, various couplings, and certain basic properties of the random walk. We hope that the development of these method will allow us to obtain also a lower bound for $\rho(t)$.

Regarding the sharpness of the estimates in (1.1a) and (1.1b), we present the following heuristics. Assume that the annihilating exclusion and the annihilating random walks start from the same configuration. Divide the space into regions such that each one contains only one type of particle. (It may be well the case that there are few or even just one particle in each region. This happens if A and B particles are properly mixed.) Consider the particles from an arbitrary region. Recall that as a consequence of Liggett's inequality it is customary to believe that at time $t > 0$ these particles will be more spread out if they interact by exclusion than if they walk independently. Thus, particles of one type will propagate into regions of another type more rapidly in the annihilating exclusion than in the annihilating random walks. Consequently, we expect $\rho(t)$ decays at least not slower than $\rho_{\text{indep}}(t)$. Unfortunately, we could not give a rigorous proof along the lines of the above reasoning. The obstacle was that even though the processes are close at time $t \approx 0$, their distributions may be quite different for $t \gg 0$. So, though we expect $\rho(t) < Ct^{-d/4}$ for $d \leqslant 4$, the technique which we used did not allow us to get rid of the ε term in (1.1a). In dimensions $d > 4$, we believe (1.1b) gives the correct exponent of the asymptotic upper bound. This belief is based on some partial results obtained in ref. 3.

2. PROOF OF THEOREM 1.1

2.1. *A formal construction of the process under consideration* and an auxiliary process will be given through the concepts of graphical representation for interacting particle systems (see ref. 6 or ref. 7, Chapter III, Section 6).

To each pair of nearest neighbor sites $x, y \in \mathbb{Z}^d$ we assign a family of independent exponential mean-$1/(2d)$ random variables $H_{xy}(l)$, $l \in \mathbb{Z}^+$, and at the random times $T_{xy}(l) := \sum_{k=1}^{l} H_{xy}(k)$, $l \in \mathbb{Z}^+$, we draw a double arrow which points to the sites x and y in the space-time diagram $\mathbb{Z}^d \times [0, \infty)$. For reasons which will become clear in a moment, we mark these arrows with A. The obtained random (directed) graph is called a *percolation substructure*. The set of all percolation substructures is denoted by Ω^A. The σ-algebra $\mathscr{F}^A$ and the measure μ^A on $\mathscr{F}^A$ are naturally inherited from the distribution of the (independent) random variables $\{H_{xy}(l), l \in \mathbb{Z}^+, x, y \in \mathbb{Z}^d, x \neq y\}$.

We will consider processes in which particles of two types participate. We call these types A and B. By $\mathscr{X} := \{A; B; A \cup B; 0\}^{\mathbb{Z}^d}$ we denote the set of all configurations of A and B particles on $\mathbb{Z}^d$ in which two or more particles of the same type are prohibited at the same site of $\mathbb{Z}^d$. We write $\eta(x) = A$, B, $A \cup B$, or 0 if the site $x \in \mathbb{Z}^d$ is occupied by respectively an A particle, a B particle, both an A and a B particle, or is empty in the configuration $\eta \in \mathscr{X}$.

Assume at time 0 a site $x \in \mathbb{Z}^d$ is occupied by an A particle. We then define the evolution of this particle on $\mathbb{Z}^d$ as the function of $\omega \in \Omega^A$ as follows: the particle stays at a site until the time when there is an arrow in ω emanating from this site. At this time, it jumps to the site pointed at by the arrow and remains there until an arrow drawn from that site appears in ω, then the procedure is repeated. By ${}^A\eta_t^x = {}^A\eta_t^x(\omega)$ we denote the position in $\omega \in \Omega^A$ at time t of the particle which occupied initially the site x. For an arbitrary $\eta_0 \in \mathscr{X}$, we denote ${}^A\eta_t := \bigcup_{x:\, \eta_0(x) = A} {}^A\eta_t^x$ and call ${}^A\eta_t$, $t \geqslant 0$, *the stirring process with the initial configuration* η_0. The distribution of this process is determined by the measure μ^A and the initial configuration. Observe that only A particles participate in ${}^A\eta_t$, $t \geqslant 0$.

Next, we repeat the procedure described two paragraphs above, independently on the same space-time diagram, but we now mark each arrow with B. Postulating that the B-type particles are moved by the B arrows, we then define another stirring process ${}^B\eta_t := \bigcup_{x:\, \eta_0(x)=B} {}^B\eta_t^x$, $t \geqslant 0$, in the same way as ${}^A\eta_t$, $t \geqslant 0$, has been defined above. Observe that these stirring processes are independent by the construction.

Finally, we define the process η_t, $t \geqslant 0$, by postulating that if $\eta_0 \in \mathscr{X}$ is the initial configuration of this process, then

$$\eta_t := \bigcup_{\eta_0(x)=A} {}^A\eta_t^x \cup \bigcup_{\eta_0(x)=B} {}^B\eta_t^x, \qquad t \geqslant 0 \tag{2.1}$$

By $\mathbb{P}_{\eta_0}$ and $\mathbb{E}_{\eta_0}$ we denote the probability law and the expectation operator governing the process η_t, $t \geqslant 0$, with the initial configuration η_0. For η_0 being given, the law $\mathbb{P}_{\eta_0}$ emanates from the measure $\mu := \mu_A \times \mu_B$ on the space $(\Omega, \mathscr{F}) := (\Omega^A \times \Omega^B, \mathscr{F}^A \times \mathscr{F}^B)$ in a natural way.

For every $\eta_0 \in \mathscr{Y} := \{A, B, 0\}^{\mathbb{Z}^d}$ and every $\omega \in \Omega$, we define the function $\tau_\cdot = \tau_\cdot(\eta_0, \omega)$ from $\mathbb{Z}^d$ to $\mathbb{R}^+$ in the following way: first, we mark each particle in η_0 "alive"; then we postulate that when an alive particle moves to a site which is currently occupied by an alive particle of the opposite type, then both change their marks from "alive" to "dead"; $\tau_x(\eta_0, \omega)$ is then defined as the death time in ω of the particle which occupied the site x in η_0; τ_x is not defined for x such that $\eta_0(x) = 0$.

Using the notation introduced above, we now define three new processes as follows: if $\xi_0 \in \mathscr{Y}$ is the initial configuration for these three processes, then we set $\eta_0 = \zeta_0$ and define

$${}^A\xi_t := \bigcup_{\substack{\eta_0(x)=A\\ \tau_x > t}} {}^A\eta_t^x, \qquad {}^B\xi_t := \bigcup_{\substack{\eta_0(x)=B\\ \tau_x > t}} {}^B\eta_t^x, \qquad \xi_t := {}^A\xi_t \cup {}^B\xi_t, \qquad t \geqslant 0 \tag{2.2}$$

The process which we have introduced in Section 1 relates to ξ_t, $t \geqslant 0$, in the following manner: starting from the same configuration, both processes have equal probabilities to belong to an arbitrary given subset of $\mathscr{Y}$ at an arbitrary given time $t \geqslant 0$. Indeed, it is possible to couple these two processes in such a way that the set of all particles of the same type in one process forms a permutation of the set of all particles of the analogous type in the other process. Since we do not distinguish between particles of the same type, we will conduct our calculation for the process ξ_t, $t \geqslant 0$, which, from here on, will be referred to as *the two-particle annihilating exclusion.*

Denote by ν the measure on $\mathscr{Y}$ which governs the construction of initial configurations as described in Section 1; we then set $\mathbb{P}(\cdot) := \int \mathbb{P}_\zeta(\cdot)\, \nu(d\zeta)$; the expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.

2.2. *A simplification of notations.* It will almost always be the case that at most one particle is present at every site of $\mathbb{Z}^d$ in the initial configurations for the processes η and ξ. In this case, we will loosely write η_t^x and ξ_t^x, omitting the left superscript, which indicates the type of the particle which originated from x.

2.3. *Two more definitions.* For every $\eta_0 \in \mathscr{Y}$, $t \geqslant 0$, and $\omega \in \Omega$, we define the function $k(\cdot) = k(\cdot; \eta_0, t, \omega)$ from $\mathbb{Z}^d$ to $\mathbb{Z}^d$ by the following rule: if $\eta_0(y) \neq 0$ and $\tau_y(\eta_0, \omega) \leqslant t$, then $k(y)$ denotes the initial position of the annihilating companion of the particle which was initially at y; if $\tau_y(\eta_0, \omega) > t$, then $k(y) := y$; if $\eta_0(y) = 0$, then $k(y)$ is not defined.

We remark that nowhere will we be interested in the value of $k(y)$ itself. Instead, the function $k(\cdot)$ is exploited to introduce the following notation, which will be frequently used in the sequel:

$$\eta_t^{k(y)} = \eta_t^{k(y)}(\omega) := \begin{cases} \eta_t^y & \text{if} \quad \tau_y(\eta_0, \omega) > t \\ \eta_t^{k(y)} & \text{if} \quad \tau_y(\eta_0, \omega) \leqslant t \end{cases} \tag{2.3}$$

In words, $\eta_t^{k(y)}$ denotes the position at time t of the annihilating companion of the particle which initiated from the site y if the annihilation has occurred before t; if, on the contrary, this particle has not been killed by t, then $\eta_t^{k(y)} = \eta_t^y$.

2.4. *The basic designations* which will be frequently used in the proofs are listed below. The second argument in (ix)–(xii) may be either ξ or η. In the case it is ξ, it usually will be omitted.

(i) W_r^w, $r \geqslant 0$, and Z_r^z, $r \geqslant 0$, denote, respectively, a simple random walk in $\mathbb{Z}$ starting from w and a simple random walk in $\mathbb{Z}^d$ starting from z.

(ii) $C, C_1, C_2,..., c_1, c_2,...,$ are absolute constants.

(iii) $t_1(\varepsilon), t_2(\varepsilon),...,$ are absolute constants which may depend on ε.

(iv) $R_t := (\delta_1 t)^{1/2}$, where $t \geqslant 0$ stands for time and the constant δ_1 will be chosen after (2.51) in accordance with β_3, the latter being an absolute constant whose value is established in the proof of Lemma 2.13.

(v) $r_t := \delta_2 t^{1/4}$ when $d < 4$, and $:= (\delta_1 t)^{1/d}$ when $d \geqslant 4$, where $t \geqslant 0$ stands for time and the constant δ_2 depends exclusively on δ_1 and is also specified in the course of the proof of Lemma 2.13.

(vi) If there is no explicit indication, D stands for a cube in $\mathbb{R}^d$ centered at the origin; to avoid complications, we always assume that the boundary of D contains no integer points of $\mathbb{R}^d$; $|D|$ is then the number of integer points contained in the cube D; ∂D is the set of the integer points of the interior and the exterior of the cube which are at the distance $\leqslant 1$ from the boundary of D.

(vii) $I_D(\eta_t^x) := 1$ if $\eta_t^x \in D$ and $:= 0$ otherwise.

(viii) D_R is a cube in $\mathbb{R}^d$ of the side R_t which is defined in (iv) above.

(ix) $\mathscr{D}_R^A(s; \cdot)$ is the number of A-type particles of the process $\cdot$ in the cube D_R at time s; $\mathscr{D}_R^B(s; \cdot)$ has the analogous meaning for B-type particles.

(x) $\mathscr{D}_R^T(s; \cdot) := \mathscr{D}_R^A(s; \cdot) + \mathscr{D}_R^B(s; \cdot)$ is the total number of particles in D_R at time s.

(xi) $\mathscr{D}_R^m(s; \cdot) := \min\{\mathscr{D}_R^A(s; \cdot), \mathscr{D}_R^B(s; \cdot)\}$ is the number of the particles which are in the minority.

(xii) $|\mathscr{D}_R(s; \cdot)| := |\mathscr{D}_R^A(s; \cdot) - \mathscr{D}_R^B(s; \cdot)|$ is the absolute value of the difference between the A- and B-type particles in D_R at time s.

2.5. *Basic properties of the constructed processes* which will be essentially used throughout the proofs are listed below.

(i) For every $x \in \mathbb{Z}^d$, ${}^A\eta_t^x$, $t \geq 0$, is a simple random walk in $\mathbb{Z}^d$ starting from x; the same is true for ${}^B\eta_t^x$, $t \geq 0$.

(ii) Let Σ and Δ be two arbitrary subsets of $\mathbb{Z}^d$ and ξ and ζ be two arbitrary configurations from $\mathscr{X}$ such that $\xi(x) = \zeta(x) = A$, $\forall x \in \Sigma$, and $\xi(x) = \zeta(x) = B$, $\forall x \in \Delta$. Then, for arbitrary $t \geq 0$ and $\omega \in \Omega$ and each $x \in \Sigma$ ($y \in \Delta$), ${}^A\eta_t^x(\omega)$ [${}^B\eta_t^y(\omega)$, respectively] attains the same value for both $\eta_0 = \xi$ and $\eta_0 = \zeta$. This fact follows from the definition of η_t, $t \geq 0$.

(iii) Let $\eta_0, \eta'_0 \in \{A, 0\}^{\mathbb{Z}^d}$ be such that $\eta'_0 \subseteq \eta_0$, which means that $\eta'_0(x) = A \Rightarrow \eta_0(x) = A$, $\forall x \in \mathbb{Z}^d$. Then

$$\bigcup_{x:\, \eta'_0(x) = A} {}^A\eta_t(\omega) \subseteq \bigcup_{x:\, \eta_0(x) = A} {}^A\eta_t(\omega) \subseteq \eta_t(\omega)$$

$$\text{for all } t \geq 0 \text{ and all } \omega \in \Omega^A$$

This property is called additivity. It can be proved using (ii) above.

Lemma 2.1. Let D be an arbitrary finite subset of $\mathbb{Z}^d$. For all $s \geq 0$, it holds that

$$E_1(s) := \mathbb{E}\left(\sum_{x:\, \eta_0(x) = A} I_D(\eta_s^x) - \sum_{u:\, \eta_0(u) = B} I_D(\eta_s^u) \right)^2 = 2\rho(0)\,|D|$$

$2\rho(0)$ is the density of the initial measure.

Proof. ${}^A\eta_.$ and ${}^B\eta_.$ evolve in time independently of one another, each one according to the rules of the (symmetric simple) exclusion process, and since both of them start from the measure which is invariant for this process (see ref. 7 for the proof that the Bernoulli product measure with a constant density is invariant for the symmetric simple exclusion process), then $E_1(s) = E_1(0)$ for all $s \geq 0$. But, due to the construction of the initial measure, $\#\{x \in D: \eta_0^x = A\} - \#\{u \in D: \eta_0^u = B\}$ is equal to the sum of $|D|$

independent random variables each of which takes on the values 1 or -1 with probability $\rho(0)$, or the value 0 with probability $1-2\rho(0)$. Since the variance of the sum of these variables is $E_1(s)$, then the latter is equal to $2\rho(0)\,|D|$. ∎

Lemma 2.2. For an arbitrary D, a finite subset of $\mathbb{Z}^d$, $\eta_0 \in \mathscr{Y}$, and distinct $x, y \in \mathbb{Z}^d$ such that $\eta_0(x) = \eta_0(y) = A$, it holds that

$$\mathbb{E}_{\eta_0}[(I_D(\eta_s^x) - I_D(\eta_s^{k(x)}))(I_D(\eta_s^y) - I_D(\eta_s^{k(y)}))] \leqslant 0 \qquad \text{for all} \quad s \geqslant 0 \tag{2.4}$$

Proof. Let D be arbitrary but fixed. Let $a, b, c \in \mathbb{Z}^d$ be arbitrary but such that a and b are not necessarily distinct, while $c \neq a$, $c \neq b$. Construct the configuration ζ by setting

$$\zeta(a) = A, \qquad \zeta(b) = B, \qquad \zeta(c) = A \cup B, \qquad \zeta(z) = 0 \quad \forall z \notin \{a; b; c\} \tag{2.5}$$

By Liggett's inequality (ref. 7, Lemma 4.12, Chapter VIII), for all $r \geqslant 0$,

$$\begin{aligned} &\mathbb{E}_\zeta[I_D({}^A\eta_r^a)\, I_D({}^A\eta_r^c)] \leqslant \mathbb{E}_\zeta[I_D({}^A\eta_r^a)]\; \mathbb{E}_\zeta[I_D({}^A\eta_r^c)] \\ &\mathbb{E}_\zeta[I_D({}^B\eta_r^b)\, I_D({}^B\eta_r^c)] \leqslant \mathbb{E}_\zeta[I_D({}^B\eta_r^b)]\; \mathbb{E}_\zeta[I_D({}^B\eta_r^c)] \end{aligned} \tag{2.6}$$

It follows from the construction of Section 2.1 that if the process η_t, $t \geqslant 0$, starts from the configuration ζ, then for all $r \geqslant 0$, ${}^A\eta_r^a$ is independent of ${}^B\eta_r^c$, ${}^A\eta_r^c$ is independent of ${}^B\eta_r^b$, and $I_D({}^A\eta_r^c)$ and $I_D({}^B\eta_r^c)$ are identically distributed. Together with (2.6) this gives that

$$\mathbb{E}_\zeta[(I_D({}^A\eta_r^a) - I_D({}^B\eta_r^b))(I_D({}^A\eta_r^c) - I_D({}^B\eta_r^c))] \leqslant 0 \tag{2.7}$$

for all $r \geqslant 0$ and independently of our choice of the sites a, b, and c.

Let now η_0 and $x, y \in \mathbb{Z}^d$ be arbitrarily fixed and satisfy the condition of the lemma. It follows from definition (2.3) that

$$[I_D(\eta_s^x) - I_D(\eta_s^{k(x)})][I_D(\eta_s^y) - I_D(\eta_s^{k(y)})] = 0$$

on those realizations of the process η_t, $t \geqslant 0$, for which $\max(\tau_x, \tau_y) \geqslant s$. In the rest of the proof, we consider only those realizations for which $\tau_x \leqslant \tau_y \leqslant s$. The case $\tau_y \leqslant \tau_x \leqslant s$ may be treated in the same way. By the strong Markov property and the property (ii) of Section 2.5 for the process η_t, $t \geqslant 0$,

$$\begin{aligned} &\mathbb{E}_{\eta_0}[I\{\tau_x \leqslant \tau_y \leqslant s\}(I_D(\eta_s^x) - I_D(\eta_s^{k(x)}))(I_D(\eta_s^y) - I_D(\eta_s^{k(y)})) \mid \tau_y, \eta_{\tau_y}] \\ &\quad = \begin{cases} 0 & \text{when} \quad \tau_y > s \quad \text{or} \quad \tau_x > \tau_y \\ \mathbb{E}_\zeta[(I_D({}^A\eta_{s-\tau_y}^a) - I_D({}^B\eta_{s-\tau_y}^b))(I_D({}^A\eta_{s-\tau_y}^c) - I_D({}^B\eta_{s-\tau_y}^c))] & \\ \quad \text{otherwise} & \end{cases} \end{aligned} \tag{2.8}$$

where

$$a = {}^{A}\eta_{\tau_y}^{x}, \qquad b = {}^{B}\eta_{\tau_y}^{k(x)}, \qquad c = {}^{A}\eta_{\tau_y}^{y} = {}^{B}\eta_{\tau_y}^{k(y)}$$

and ζ is constructed by (2.5). From (2.7) we conclude that the second line of the right-hand side of (2.8) is $\leqslant 0$ for all τ_y and all η_{τ_y}. Consequently, the left-hand side of (2.8) is $\leqslant 0$ for all τ_y and all η_{τ_y}. Taking mathematical expectation validates the lemma's assertion. ▮

The following lemma is crucial for the proof of Theorem 1.1. In this lemma, we consider two processes. The initial configuration for both is the same. In one process, the particles execute a stirring process. In the other, we exclude one particle from stirring with the others and postulate that it executes an independent random walk. We are interested in comparing $V(t) f(\chi; v)$ to $U(t) f(\chi; v)$, where $V(\cdot)$, $U(\cdot)$ are the semigroups of these processes, f is a particular function of the form (2.9) below, v is the initial position of the excluded particle, and χ is the initial configuration of the rest of the particles. The comparison will be done utilizing the same technique as one uses to prove Liggett's inequality, the one to which we referred when proving (2.6). We remark that this inequality gives just a qualitative estimate, i.e., $V(t) f(\chi; v) - U(t) f(\chi; v) \leqslant 0$. The work of Andjel[1] drew our attention to the fact that, using the same technique, one can evaluate the above difference quantitively.

To formulate and prove the lemma, we start with some auxiliary constructions. The connection of the annihilating exclusion process to these constructions will become clear in the course of the proof of Lemma 2.5.

Let $\hat{X} = \{0, 1\}^{\mathbb{Z}^d} \times \{0, 1\}$ be the set of all the configurations of particles on $\mathbb{Z}^d$ in which (i) one particle is marked, (ii) no more than one unmarked particle is allowed at a site, (iii) the marked particle may occupy a site which contains an unmarked particle. By $(\chi; y)$ we denote the configuration from $\hat{X}$ in which the marked particle stays at the site y and the unmarked particles form the configuration $\chi \in \{0, 1\}^{\mathbb{Z}^d}$. For a configuration χ and two sites $u, v \in \mathbb{Z}^d$ such that $\chi(u) = 1$, $\chi(v) = 0$, we denote by $\chi(u, v)$ the configuration which is obtained from χ by moving the particle from u to v.

By $U(t)$, $t \geqslant 0$, we denote the semigroup of the process on $\hat{X}$ in which the unmarked particles evolve according to the stirring mechanism and the marked particle executes an independent random walk. By $V(t)$, $t \geqslant 0$, we denote the semigroup of the process on $\bar{X} := \{(\phi; y) \in \hat{X}\colon \phi(y) = 0\}$ in which all the particles evolve according to the stirring mechanism. Let U and V designate the generators of these two processes.

From here on, the symbol f will denote the function from $\hat{X}$ to $\mathbb{R}$ which is defined by

$$f(\chi; y) := \left(\sum_{x:\, \chi(x)=1} I_D(x) \right) I_D(y), \qquad (\chi; y) \in \hat{X} \tag{2.9}$$

where D is a finite subset of $\mathbb{Z}^d$ which will be explicitly specified every time we refer to this function.

Lemma 2.3. For an arbitrary fixed $h \in \mathbb{N}$, let D be the cube in $\mathbb{Z}^d$ with the side $2h+1$ centered at the origin, i.e., $D = [-h, h]^d \cap \mathbb{Z}^d$, and denote by H the set $\{-h; -h-1; h; h+1\}$. Let $(\chi; v)$ be an arbitrary configuration from $\bar{X}$. Then, for f defined by (2.9) it holds that

$$0 \leqslant [U(s) - V(s) f](\chi; v) \leqslant \frac{1}{d} \sum_{j=1}^{d} \int_0^s \sum_{y \in \mathbb{Z}} P[W_{s-r}^{v^j} = y](P[W_r^y \in H])^2 \, dr \tag{2.10}$$

where v^j is the jth coordinate of the point v in $\mathbb{R}^d$ and W is defined in (i) of Section 2.4.

Proof. Fix arbitrary χ from $\{0, 1\}^{\mathbb{Z}^d}$ and $v \in \mathbb{Z}^d$ such that $\chi(v) = 0$. From the integration by part formula,

$$[U(s) - V(s) f](\chi; v) = \left\{ \int_0^s V(s-r)[U - V] \, U(r) g \, dr \right\} (\chi; v) \tag{2.11}$$

It is straightforward to check that for a function $g: \hat{X} \to \mathbb{R}$ and for a configuration $(\phi; y) \in \bar{X}$, it holds that ($x \sim y$ means that these sites are neighbors on $\mathbb{Z}^d$)

$$\begin{aligned}
([U - V] g)(\phi; y) = {} & \sum_{x:\, \phi(x)=1,\, x \sim y} \frac{1}{2d} [g((\phi(x, y); y)) - g(\phi; y)] \\
& + \sum_{x:\, \phi(x)=1,\, x \sim y} \frac{1}{2d} [g(\phi; x) - g(\phi; y)] \\
& - \sum_{x:\, \phi(x)=1,\, x \sim y} \frac{1}{2d} [g((\phi(x, y); x)) - g(\phi; y)] \\
= {} & \sum_{x:\, \phi(x)=1,\, x \sim y} \frac{1}{2d} [g((\phi(x, y); y)) - g(\phi; y) \\
& + g((\phi; x)) - g((\phi(x, y); x))]
\end{aligned} \tag{2.12}$$

[Certainly, (2.12) may be wrong when g is not in the domains of definition of U and V. In what follows, however, g will be taken equal to $U(r) f$. Since f is in both domains of definition of the generators U and V, then so is g due to the Hille–Yosida theorem.[7]]

Let $e_j := (0,..., 0, 1, 0,..., 0)$ be the point in $\mathbb{R}^d$ whose jth coordinate is 1. We define a family of operators $\{F_j^+, F_j^-, j = 1,..., d\}$ on the set of the functions from $\hat{X}$ to $\mathbb{R}$ by

$$F_j^\pm g(\phi; y) := \frac{1}{2d} [g((\phi(y \pm e_j, y); y)) - g(\phi; y) + g((\phi; y \pm e_j)) \\ - g((\phi(y \pm e_j, y); y \pm e_j))]$$

iff $\phi(y \pm e_j) = 1$, and $F_j^\pm g(\phi; y) := 0$ otherwise. Using (2.12) and the functions $F_j^\pm$, we rewrite (2.11) in the following equivalent form:

$$\sum_{j=1}^{d} \left\{ \int_0^s V(s-r)[F_j^+ + F_j^-] \, U(r) f \, dr \right\} (\chi; v) \tag{2.13}$$

Our current objective is to evaluate (2.13) based on the properties of a simple random walk and of a stirring process. The technique which we will use is called coupling. To couple processes, we will use an independent copy of the probability space of percolation substructures $(\Omega^A, \mathscr{F}^A, \mu_A)$ which has been defined in Section 2.1. This probability space generates the stirring process in the way explained in Section 2.1. Also, by $(\Gamma, \mathscr{F}^\Gamma, \mu_\Gamma)$ we will denote the probability space of the percolation substructures which generates the simple random walk on $\mathbb{Z}^d$ which starts at zero. We postulate that the stirring process and the random walk generated by the probability spaces above are independent.

Assume $\omega \in \Omega$, $\gamma \in \Gamma$, and $r \geqslant 0$. Put a configuration $\phi \in \{0, 1\}^{\mathbb{Z}^d}$ at time 0 in ω. The configuration which is obtained from ϕ at time r in ω will be designated by ϕ_r^ω. Similarly, put a particle at time zero at the origin in γ. The position of this particle at time r in γ will be designated by 0_r^γ. For $y \in \mathbb{Z}^d$, we then define $y_r^\gamma := 0_r^\gamma + y$, so that y_r^γ, $r \geqslant 0$, is a trajectory of a simple random walk in $\mathbb{Z}^d$ starting from y. Also, by $(\phi; y)_r^{\omega,\gamma}$, we will designate the configuration $(\varphi; z) \in \hat{X}$ such that $\varphi = \phi_r^\omega$ and $z = y_r^\gamma$.

Denote

$$\pm\partial_1 D := \{z = (z^1,..., z^d) \in D\colon z^1 = \pm h\}$$
$$-\partial_1 D - e_1 := \{x - e_1 : x \in -\partial_1 D\}$$

Let now $(\phi; y)$ be an arbitrarily fixed configuration from $\bar{X}$ such that $\phi(y + e_1) = 1$ and denote $x := y + e_1$. The idea of introducing $\partial_1 D$ and $-\partial_1 D - e_1$ lies in the following equivalences, which can be easily checked:

$$\gamma \in \Gamma \text{ is such that } y_r^\gamma \in D \text{ and } x_r^\gamma \notin D \\ \Leftrightarrow \gamma \in \Gamma \text{ is such that } y_r^\gamma \in \partial_1 D \tag{2.14}$$

$$\gamma \in \Gamma \text{ is such that } y_r^\gamma \notin D \text{ and } x_r^\gamma \in D$$
$$\Leftrightarrow \gamma \in \Gamma \text{ is such that } y_r^\gamma \in -\partial_1 D - e_1 \tag{2.15}$$

For $\gamma \in \Gamma$, $\omega \in \Omega$, $r \geqslant 0$, and the configuration $(\phi; y)$ fixed above, consider

$$f[(\phi(x, y); y)_r^{\omega,\gamma}] - f[(\phi; y)_r^{\omega,\gamma}] + f[(\phi; x)_r^{\omega,\gamma}] - f[(\phi(x, y); x)_r^{\omega,\gamma}] \tag{2.16}$$

Consider (2.16) for γ such that y_r^γ and x_r^γ either both belong to D or both do not belong to it. Then, for every ω, the first summand cancels the fourth summand and the second one cancels the third one. Thus, for the considered fixed γ, the mathematical expectation over ω with respect to the measure μ_A (which we denote by E_Ω) of (2.16) is zero.

Assume that $\gamma \in \Gamma$ satisfies (2.14). Then, for this γ and for every $\omega \in \Omega$, the last two terms of (2.16) are zeros. Furthermore, for an arbitrary ω, consider the configurations $(\phi(x, y))_r^\omega$ and $(\phi)_r^\omega$. They differ at two sites, which are the position in $(\phi(x, y))_r^\omega$ of the particle which started at the site y and the position in $(\phi)_r^\omega$ of the particle which started at x. Consequently, for γ which satisfies (2.14), the mathematical expectation with respect to the measure μ_A of (2.16) equals

$$E_\Omega(f[(\phi(x, y); y)_r^{\omega,\gamma}] - f[(\phi; y)_r^{\omega,\gamma}]) = E_\Omega(I_D(y_r^\omega) - I_D(x_r^\omega)) \tag{2.17}$$

But, due to (i) of Section 2.5, $y_r^\cdot$, $r \geqslant 0$, and $x_r^\cdot$, $r \geqslant 0$, are two random walks in $\mathbb{Z}^d$ that start from two neighboring points x and y such that $x = y + e_1$. Thus, coupling these random walks in such a way that they use simultaneously the same percolation substructure from Γ and using (2.14) and (2.15), we derive that (2.17) equals (γ is a random variable from Γ and not that which has been fixed above, and by E_Γ we denote the mathematical expectation with respect to the measure μ_Γ)

$$E_\Gamma[I_D(y_r^\gamma) - I_D(x_r^\gamma)] = P[Z_r^y = \partial_1 D] - P[Z_r^y \in -\partial_1 D - e_1] \tag{2.18}$$

Also, (2.14) gives that

$$\mu_\Gamma[\gamma \in \Gamma\colon y_r^\gamma \in D \text{ and } x_r^\gamma \notin D] = P[Z_r^y \in \partial_1 D] \tag{2.19}$$

From (2.18) and (2.19) we derive that the $E_{\Omega \times \Gamma}$ mathematical expectation of (2.16) over those $\gamma \in \Gamma$ which satisfy (2.14) is

$$P[Z_r^y \in \partial_1 D](P[Z_r^y \in \partial_1 D] - P[Z_r^y \in -\partial_1 D - e_1])$$

Similar reasoning gives that the $E_{\Omega \times \Gamma}$ mathematical expectation of (2.16) over those $\gamma \in \Gamma$ which satisfy (2.15) is

$$P[Z_r^y \in -\partial_1 D - e_1](P[Z_r^y \in -\partial_1 D - e_1] - P[Z_r^y \in \partial_1 D])$$

Consequently, the $E_{\Omega \times \Gamma}$ mathematical expectation of (2.16) equals

$$\begin{cases} (P[Z_r^y \in -\partial_1 D - e_1] - P[Z_r^y \in \partial_1 D])^2 & \text{if } \; y + e_1 \in \phi \\ 0 & \text{otherwise} \end{cases} \tag{2.20}$$

[the second line in (2.20) follows immediately from the definition of F_1^+]. It is important to observe that, as follows from our reasoning, the $E_{\Omega \times \Gamma}$ mathematical expectation of (2.16) depends exclusively on the value of ϕ at the site $y + e_1$ as specified in (2.20). Using this observation, we conclude from (2.20) that

$$0 \leqslant F_1^+ U(r) f(\phi; y) \leqslant \frac{1}{2d} (P[Z_r^y \in -\partial_1 D - e_1] - P[Z_r^y \in \partial_1 D])^2 \tag{2.21}$$

independently of ϕ. Now use once more the fact that the marginal motion of a particle in the stirring process is a simple random walk [(i) of Section 2.5]. Thus, from the above formula we get that

$$\begin{aligned} 0 &\leqslant V(s-r) F_1^+ U(r) f(\xi; v) \\ &\leqslant \frac{1}{2d} \sum_{y \in \mathbb{Z}^d} P[Z_{s-r}^v = y] \\ &\quad \times (P[Z_r^y \in -\partial_1 D - e_1] - P[Z_r^y \in \partial_1 D])^2 \\ &\leqslant \frac{1}{2d} \sum_{y \in \mathbb{Z}} P[W_{s-r}^{v^1} = y](P[W_r^y \in H])^2 \end{aligned} \tag{2.22}$$

Observe that the independence of ϕ in (2.21) is essentially used in deriving the second inequality in (2.22); the third inequality is an obvious consequence of the properties of random walks.

Reasoning in the same way, one derives that for every $j = 1, 2,..., d$, $V(s-r) F_j^{\pm} U(r) f(\xi; v)$ is not less than 0 and not greater than the right-hand side of (2.22) with v^1 being changed for v^j. Together with (2.13), this gives (2.10). ∎

The following lemma estimates the integral in (2.10).

Lemma 2.4. Let H be an arbitrary finite subset of $\mathbb{Z}$ consisting of $|H|$ sites and let x be an arbitrary site in $\mathbb{Z}$. Then,

$$\int_0^s \sum_{z \in \mathbb{Z}} P[W_{s-r}^x = z](P[W_r^z \in H])^2 \, dr \leqslant C_1 \, |H| \tag{2.23}$$

where C_1 is an appropriate absolute constant.

Proof. Recall that for a simple random walk in $\mathbb{Z}^d$,

$$\sup_{y \in \mathbb{Z}^d} P[Z_r^x = y] \leqslant \frac{C_2}{(r)^{d/2}} \qquad \text{for all} \quad x \in \mathbb{Z}^d \text{ and all } r \geqslant 0 \tag{2.24}$$

Thus, using (2.24) and the Markov property of a simple random walk, we have that the left-hand side of (2.23) is

$$\begin{aligned} &\leqslant \int_0^s \frac{C_2}{(r)^{1/2}} \sum_{z \in \mathbb{Z}} P[W_{s-r}^x = z]\, P[W_r^z \in H]\, dr \\ &\leqslant \int_0^s \frac{C_2}{(r)^{1/2}} \frac{C_2\, |H|}{(s)^{1/2}}\, dr \leqslant C_1\, |H| \end{aligned} \tag{2.25}$$

which establishes the assertion of the lemma. ∎

The previous two lemmas are combined together to obtain the following result.

Lemma 2.5. Let R_t be given by (iv) of Section 2.4 and let $D = D_R$ be the cube in $\mathbb{R}^d$ with the side R_t centered at the origin. For every $\varepsilon > 0$, there exists $t_2(\varepsilon)$ such that for all $t \geqslant t_2(\varepsilon)$ and all $s \in [t/2; t]$

$$\mathbb{E}\Bigg[\sum_{\substack{x,\, y:\, x \neq y \\ \eta_0(x) = \eta_0(y) = A}} I_D(\eta_s^x)(I_D(\eta_s^{k(y)}) - I_D(\eta_s^y))\Bigg] \leqslant C_5 R_t^d t^{\varepsilon} \tag{2.26}$$

where C_5 is an appropriate absolute constant.

Proof. Let y be an arbitrary site of $\mathbb{Z}^d$ and $\eta_0 \in \mathscr{Y}$ be an arbitrary configuration such that $\eta_0(y) = A$. The strong Markov property and the definition of $k(\cdot)$ give that

$$\begin{aligned} &\mathbb{E}_{\eta_0}\Bigg[\sum_{\substack{x:\, x \neq y \\ \eta_0(x) = A}} I_D(\eta_s^x)(I_D(\eta_s^{k(y)}) - I_D(\eta_s^y)) \,\Bigg|\, \tau_y, \eta_{\tau_y}\Bigg] \\ &= \begin{cases} \mathbb{E}_\zeta\Bigg[\displaystyle\sum_{\substack{x:\, x \neq v \\ \zeta(x) = A}} I_D(\eta_{s-\tau}^x)(I_D({}^B\eta_{s-\tau}^v) - I_D({}^A\eta_{s-\tau}^v))\Bigg] & \text{if} \quad \tau_y \leqslant s \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{2.27}$$

where in order to avoid complicated indexes, we denoted $\tau = \tau_y$, $\zeta = \eta_{\tau_y}$, and $v = \eta_{\tau_y}^y = \eta_{\tau_y}^{k(y)}$. According to the constructions which precede Lemma 2.3, the upper line of the right-hand side of (2.27) equals

$$([U(s-\tau) - V(s-\tau)]f)(\chi; v) \tag{2.28}$$

with f being defined by (2.9) for $D=D_R$ and χ being the set of all sites of $\mathbb{Z}^d\backslash\{v\}$ which are occupied by A-type particles in ζ.

Due to Lemmas 2.3 and 2.4, the expression (2.28) is less than $C_3 := C_1\,|H| = 4C_1$ for all $s \geqslant 0$, $\tau_y \leqslant s$, and $(\chi; v) \in \bar{X}$. Having this fact in mind and taking the mathematical expectation of both sides of (2.27) gives

$$\mathbb{E}_{\eta_0}\left[\sum_{\substack{x:\,x\neq y\\ \eta_0(x)=A}} I_D(\eta_s^x)(I_D(\eta_s^{k(y)}) - I_D(\eta_s^y))\right] \leqslant C_3 \tag{2.29}$$

for all $\eta_0 \in \mathscr{Y}$, $s \geqslant 0$, and y such that $\eta_0(y)=A$. From (2.29) we conclude that

$$\begin{aligned}\sum_{\|y\|\leqslant R_t t^{\varepsilon}} \mathbb{E}&\left[\sum_{\substack{x:\,x\neq y\\ \eta_0(x)=\eta_0(y)=A}} I_D(\eta_s^x)(I_D(\eta_s^{k(y)}) - I_D(\eta_s^y))\right]\\ &\leqslant C_3(R_t t^{\varepsilon})^d, \qquad \forall s \in [t/2, t]\end{aligned} \tag{2.30}$$

where $\|x\| \leqslant C$ if and only if the point x belongs to the cube with the side C centered at the origin.

Consider now an A particle in an arbitrary configuration η_0. Let y denote the site which it occupies. Observe that the marginal motions of this particle and its annihilating companion after the time of their annihilation are simple random walks which start from the same point. Thus,

$$\mathbb{E}_{\eta_{\tau(y)}} I_D({}^B\eta^{k(y)}_{\tau(y)+r}) = \mathbb{E}_{\eta_{\tau(y)}} I_D({}^A\eta^{y}_{\tau(y)+r}), \qquad \forall r \geqslant 0$$

where $\tau(y)$ is substituted for τ_y to avoid complicated indexes. Using the above relation and the strong Markov property for the process η_t, $t \geqslant 0$, one easily finds that

$$\mathbb{E}_{\eta_0}(I_D({}^B\eta_s^{k(y)})\, I\{\tau_y \leqslant s\}) \leqslant \mathbb{E}_{\eta_0} I_D({}^A\eta_s^y) \tag{2.31}$$

Now, using (2.31), we can write

$$\begin{aligned}\sum_{\|y\|\geqslant R_t t^{\varepsilon}} \mathbb{E}&\left[\sum_{\substack{y,x:\,x\neq y\\ \eta_0(x)=\eta_0(y)=A}} I_D(\eta_s^x)(I_D(\eta_s^{k(y)}) - I_D(\eta_s^y))\right]\\ &\leqslant |D| \sum_{\|y\|\geqslant R_t t^{\varepsilon}} \mathbb{E}[I_D(\eta_s^{k(y)})\, I\{\tau_y < s\}\, I\{\eta_0(y)=A\}]\\ &\leqslant |D| \sum_{\|y\|\geqslant R_t t^{\varepsilon}} \mathbb{E}[I_D(\eta_s^y)]\end{aligned} \tag{2.32}$$

The central limit theorem gives that for every $\varepsilon>0$ there is $t_1(\varepsilon)$ such that

$$\sum_{x:\, \|x-y\|\geqslant r^{1/2+\varepsilon}} P[Z_r^x=y]\leqslant C_4\exp(-c_3 r^\varepsilon) \tag{2.33}$$

for all $y\in\mathbb{Z}^d$ and all $r>t_1(\varepsilon)$. Recall that D is the cube with the side $R_t=(\delta_1 t)^{1/2}$ centered at the origin. Thus, from (2.33) and the property (i) of Section 2.5, we conclude that the last expression in (2.32) is

$$\leqslant (|D|)^2\, C_4\exp[-c_3(t/2)^\varepsilon]\leqslant(\delta_1 t)^d\, C_4\exp[-c_3(t/2)^\varepsilon]\leqslant 1 \tag{2.34}$$

for all $s\in[t/2,t]$ when t is larger than an appropriate $t_2(\varepsilon)$. Combining the inequalities (2.32)–(2.34) with (2.30), we obtain the lemma's assertion. ∎

Lemma 2.6. For every $\varepsilon>0$ there exists $t_2(\varepsilon)$ such that for all $t\geqslant t_2(\varepsilon)$,

$$\mathbb{E}[|\mathscr{D}_R(s)|]\leqslant C_6 R_t^{d/2} t^\varepsilon \qquad \text{for all} \quad s\in[t/2,t]$$

where C_6 is an absolute constant.

Proof. Fix t and let D be the cube with the side R_t. Let $\eta_0\in\mathscr{Y}$ be arbitrary and fixed. It follows from the construction of Section 2.1 that

$$\begin{aligned}
&\sum_{x:\,\eta_0(x)=A} I_D(\xi_s^x)-\sum_{u:\,\eta_0(u)=B} I_D(\xi_s^u)\\
&\quad=\sum_{x:\,\eta_0(x)=A}(I_D(\eta_s^x)-I_D(\eta_s^x)\, I\{\tau_x\leqslant s\})\\
&\qquad-\sum_{u:\,\eta_0(u)=B}(I_D(\eta_s^u)-I_D(\eta_s^u)\, I\{\tau_u\leqslant s\})
\end{aligned} \tag{2.35}$$

for all $s\geqslant 0$. From here on until the end of the proof of this lemma, we will write η_s^x for $I_D(\eta_s^x)$ and ξ_s^x for $I_D(\xi_s^x)$, respectively. From (2.35) one easily derives the following relationship:

$$\begin{aligned}
&\left(\sum_{x:\,\eta_0(x)=A}\xi_s^x-\sum_{u:\,\eta_0(u)=B}\xi_s^u\right)^2\\
&\quad=\left(\sum_{x:\,\eta_0(x)=A}\eta_s^x-\sum_{u:\,\eta_0(u)=B}\eta_s^u\right)^2\\
&\qquad+\left(\sum_{y:\,\eta_0(y)=A}\eta_s^y I\{\tau_y\leqslant s\}-\sum_{v:\,\eta_0(v)=B}\eta_s^v I\{\tau_v\leqslant s\}\right)^2\\
&\qquad+2\left(\sum_{x:\,\eta_0(x)=A}\eta_s^x-\sum_{u:\,\eta_0(u)=B}\eta_s^u\right)\\
&\qquad\times\left(\sum_{v:\,\eta_0(v)=B}\eta_s^v I\{\tau_v\leqslant s\}-\sum_{y:\,\eta_0(y)=A}\eta_s^y I\{\tau_y\leqslant s\}\right)
\end{aligned} \tag{2.36}$$

Since any annihilation involves two particles of opposite types, then, using the notation (2.3), we rewrite the second summand in the right-hand side of (2.36) as

$$\sum_{y:\,\eta_0(y)=A}\ \sum_{\substack{x:\,\eta_0(x)=A\\ x\neq y}} (\eta_s^y-\eta_s^{k(y)})(\eta_s^x-\eta_s^{k(x)})+\sum_{y:\,\eta_0(y)=A}(\eta_s^y-\eta_s^{k(y)})^2 \tag{2.37}$$

and the third summand of the right-hand side of (2.36) as

$$\begin{aligned}2&\sum_{y:\,\eta_0(y)=A}\ \sum_{\substack{x:\,\eta_0(x)=A\\ x\neq y}} \eta_s^x(\eta_s^{k(y)}-\eta_s^y)+2\sum_{y:\,\eta_0(y)=A}\eta_s^y(\eta_s^{k(y)}-\eta_s^y)\\ &+2\sum_{y:\,\eta_0(y)=B}\ \sum_{\substack{x:\,\eta_0(x)=B\\ x\neq y}} \eta_s^x(\eta_s^{k(y)}-\eta_s^y)+2\sum_{y:\,\eta_0(y)=B}\eta_s^y(\eta_s^{k(y)}-\eta_s^y)\end{aligned} \tag{2.38}$$

Let us now make two observations.

(i) Due to the symmetry between A and B particles, the mathematical expectation $\mathbb{E}$ of the double sum of the first line in (2.38) equals that of the double sum of the second line in (2.38).

(ii) Due to the definition of the process η_t, $t\geqslant 0$, and Section 2.3, we have that η_s^y and $\eta_s^{k(y)}$ are identically distributed for all $s\geqslant 0$. Consequently, for an arbitrary given η_0 and y such that $\eta_0(y)=A$ we have

$$\mathbb{E}_{\eta_0}(\eta_s^y-\eta_s^{k(y)})^2+4\mathbb{E}_{\eta_0}\eta_s^y(\eta_s^{k(y)}-\eta_s^y)=-\mathbb{E}_{\eta_0}(\eta_s^y-\eta_s^{k(y)})^2$$

From the above observations and from (2.36)–(2.38) we derive that

$$\begin{aligned}\mathbb{E}&\left(\sum_{x:\,\eta_0(x)=A}\xi_s^x-\sum_{u:\,\eta_0(u)=B}\xi_s^u\right)^2\\ &=\mathbb{E}\left(\sum_{x:\,\eta_0(x)=A}\eta_s^x-\sum_{u:\,\eta_0(u)=B}\eta_s^u\right)^2\\ &\quad+\mathbb{E}\left(\sum_{y:\,\eta_0(y)=A}\ \sum_{\substack{x:\,\eta_0(x)=A\\ x\neq y}}(\eta_s^y-\eta_s^{k(y)})(\eta_s^x-\eta_s^{k(x)})\right)\\ &\quad-\mathbb{E}\left(\sum_{y:\,\eta_0(y)=A}(\eta_s^y-\eta_s^{k(y)})^2\right)\\ &\quad+4\mathbb{E}\left(\sum_{y:\,\eta_0(y)=A}\ \sum_{\substack{x:\,\eta_0(x)=A\\ x\neq y}}\eta_s^x(\eta_s^{k(y)}-\eta_s^y)\right)\end{aligned} \tag{2.39}$$

Consider the right-hand side of (2.39). The first mathematical expectation equals $2\rho(0)\, R_t^d$ due to Lemma 2.1, the second one is not positive due to Lemma 2.2, and the last one is less than $C_5 R_t^d t^\varepsilon$ due to Lemma 2.5 for s and t which satisfy the conditions of this lemma. Consequently, the left-hand side of (2.39) is bounded from above by $5C_5 R_t^d t^\varepsilon$ when $s \in [t/2, t]$ and t is larger than $t_2(\varepsilon)$. Applying the Jensen inequality $\mathbb{E}\,|(\cdot)| \leqslant \{\mathbb{E}[(\cdot)^2]\}^{1/2}$ and recalling the definition of $|\mathscr{D}_R(\cdot)|$ given in (xii) of Section 2.4, we obtain the lemma's assertion. ▮

Remark 2.7. In what follows, we will usually refer the reader to Bramson and Lebowitz (BL).[4] Thus, when we say that "a part of a proof coincides with a certain part in BL," we mean that the reader should refer to the indicated part of BL, remembering η_t and ζ_t are here defined by (2.1) and (2.2). We recall that ζ_t, $t \geqslant 0$, in BL denotes two-particle annihilating random walks, and η_t, $t \geqslant 0$, stands for the process in which the particles which are present in η_0 execute independent random walks without interactions. Except for η_t and ζ_t, all other designations have the same meaning here and in BL, though the values of the absolute constants denoted by the same notation may differ.

Definition 2.8. We define

$$h_d(T) := \min_{\|x\| \leqslant T} [\Pr\{\text{a rate-2 random walk on } \mathbb{Z}^d \text{ starting at } x \text{ hits } 0 \text{ before time } T^2\}]$$

where the norm $\|\cdot\|$ has been defined after (2.30).

The following asymptotics has been established in Lemma 4.1, BL:

$$\begin{aligned} h_d(T) &\geqslant c_1, & d&=1 \\ &\geqslant c_1/\log T, & d&=2 \\ &\geqslant c_1 T^{2-d}, & d&\geqslant 3 \end{aligned} \tag{2.40}$$

as $T \to \infty$, where $c_1 > 0$ is an absolute constant depending on d.

Lemma 2.9 (Counterpart of Lemma 4.2, BL). Let R_t be given by (iv) of Section 2.4. Assume that ξ_0 has translation-invariant initial distribution; then

$$\mathbb{E}[\mathscr{D}_R^T(s)] - \mathbb{E}[\mathscr{D}_R^T(s + R_t^2)] \geqslant h_d(R_t)\, \mathbb{E}[\mathscr{D}_R^m(s)]$$

for all s when t is large enough.

Proof. As noticed in (i) of Section 2.5, ${}^A\eta_t^x$ and ${}^B\eta_t^y$, $t \geqslant 0$, are two random walks. By the construction, they are independent. Set $\eta_0 := \xi_0$.

Then, for every pair $x, y \in \mathbb{Z}^d$ such that $\eta_0(x) = A$, $\eta_0(y) = B$, the difference $\eta_t^x - \eta_t^y$, $t \geqslant 0$, is a rate-2 random walk in $\mathbb{Z}^d$. Having in mind this fact, one easily sees that the argument used to prove Lemma 4.2 in BL suits the two-particle annihilation exclusion. This establishes the lemma. The value of t for which the lemma holds is determined by the asymptotics (2.40). ∎

From here on, r and R stand, respectively for r_t and R_t, the latter being determined in (iv) and (v) of Section 2.4.

For $r < R$, we consider the set of nonintersecting cubes $D_{r,j}$, $j \in J$, of the side r which form a partition of D_R, i.e., $\bigcup_{j \in J} D_{r,j} = D_R$. We denote $q := (r/R)^d$. The symbols $\mathscr{D}_r^A$, $\mathscr{D}_r^B$, and $\mathscr{D}_r^m$ are defined as in (ix) and (xi) in Section 2.4 for the cube D_r.

Lemma 2.10 (Counterpart of Corollary 1 after Lemma 4.4, BL). Let $R = R_t$ be given by (iv) of Section 2.4. Suppose that $\mathbb{E}[\mathscr{D}_R^m(0; \eta)] \geqslant L_1$, where $L_1 \geqslant c_2/q$ for appropriate $c_2 > 0$ (not depending on R or r). There is an absolute constant $\beta_1 > 0$ such that

$$\mathbb{E}[\mathscr{D}_{r,j}^m(s; \eta)] \geqslant \beta_1 q L_1/8 \tag{2.41}$$

for all $j \in J$ and all $s \in [R^2/2, R^2]$ when t is large enough.

Proof. The central limit theorem guarantees the existence of two absolute constants $\bar{\beta} \geqslant \beta_1 > 0$ such that $\beta_1/R^d \leqslant \Pr\{Z_s^x = y\} \leqslant \bar{\beta}/R^d$ for all $x, y \in D_R$ and all $s \in [R^2/2, R^2]$ when t is sufficiently large. Thus, using the property (i) of Section 2.5, we have that

$$\beta_1 q \leqslant E[I_{D_r}(\eta_s^x)] \leqslant \bar{\beta} q < 1 \qquad \text{for all} \quad x \in D_R, \quad s \in [R^2/2, R^2] \tag{2.42}$$

where D_r is an arbitrary cube from the partition of D_R.

Let now $k \geqslant l > 0$ be two arbitrary integers and $x_1, \ldots, x_k$ be a set of k distinct sites in D_R. Then

$$\bigcup_{i=1}^{l} {}^A\eta_t^{x_i} \subseteq \bigcup_{i=1}^{k} {}^A\eta_t^{x_i}, \qquad \forall t \geqslant 0$$

due to (iii) of Section 2.5. Thus, $\sum_{i=1}^{l} I_{D_r}({}^A\eta_t^{x_i})$ is stochastically smaller than $\sum_{i=1}^{k} I_{D_r}({}^A\eta_t^{x_i})$, so that for the considered $x_1, \ldots, x_k$ and $l \leqslant k$, the following is valid:

$$\Pr\left\{\sum_{i=1}^{k} I_{D_r}({}^A\eta_s^{x_i}) \leqslant \beta_1 q l/2\right\}$$
$$\leqslant \Pr\left\{\sum_{i=1}^{k} I_{D_r}({}^A\eta_s^{x_i}) \leqslant \frac{1}{2} E\left[\sum_{i=1}^{l} I_{D_r}({}^A\eta_s^{x_i})\right]\right\}$$

$$
\begin{aligned}
&\leqslant \Pr\left\{\sum_{i=1}^{l} I_{D_r}({}^A\eta_s^{x_i}) - E\left[\sum_{i=1}^{l} I_{D_r}({}^A\eta_s^{x_i})\right]\right. \\
&\qquad\qquad \left.\leqslant -\frac{1}{2}E\left[\sum_{i=1}^{l} I_{D_r}({}^A\eta_s^{x_i})\right]\right\} \\
&\leqslant \Pr\left\{\left|\sum_{i=1}^{l} I_{D_r}({}^A\eta_s^{x_i}) - E\left[\sum_{i=1}^{l} I_{D_r}({}^A\eta_s^{x_i})\right]\right|\right. \\
&\qquad\qquad \left.\geqslant \frac{1}{2}E\left[\sum_{i=1}^{l} I_{D_r}({}^A\eta_s^{x_i})\right]\right\} \\
&\leqslant \frac{\operatorname{Var}(\sum_{i=1}^{l} I_{D_r}({}^A\eta_s^{x_i}))}{(\frac{1}{2}E[\sum_{i=1}^{l} I_{D_r}({}^A\eta_s^{x_i})])^2} \\
&\leqslant \frac{\sum_{i=1}^{l} \operatorname{Var} I_{D_r}({}^A\eta_s^{x_i})}{\frac{1}{4}(\beta_1 q l)^2} \\
&\leqslant \frac{l \max_{\beta_1 \leqslant \beta \leqslant \bar{\beta}} q\beta(1-q\beta)}{\frac{1}{4}(\beta_1 q l)^2}
\end{aligned}
\tag{2.43}
$$

The first and the last inequality are substantiated by (2.42), and in the last but one inequality, we used the negative correlation of

$$I_{D_r}({}^A\eta_s^{x_i}), \qquad i = 1,..., k$$

(see ref. 7, Lemma 4.12, Chapter VIII).

Observe that as a consequence of (iii) of Section 2.5, a cube D_r may contain at time s some additional particles except for those which were present initially in the cube D_R. Thus one concludes from (2.43) that there is an absolute constant $\beta = \beta(\beta_1, \bar{\beta}) > 0$ such that for all $s \in [R^2/2, R^2]$ and $k \geqslant l > 0$,

$$\mathbb{P}[\mathscr{D}_r^A(s;\eta) \leqslant \tfrac{1}{2}\beta_1 q l \mid \mathscr{D}_R^A(0;\eta) = k \geqslant l] \leqslant (lq\beta)^{-1} \tag{2.44}$$

independent of the initial positions of the A-type particles in the cube D_R. Further, since ${}^A\eta_{\cdot}$ and ${}^B\eta_{\cdot}$ evolve independently, then (2.44) yields that

$$\mathbb{P}[\mathscr{D}_r^A(s;\eta) \geqslant \tfrac{1}{2}\beta_1 q l, \mathscr{D}_r^B(s;\eta) \geqslant \tfrac{1}{2}\beta_1 q l \mid \mathscr{D}_R^m(0;\eta) = k \geqslant l] \geqslant [1 - (lq\beta)^{-1}]^2$$

Recall that β depends exclusively on $\bar{\beta}$ and β_1; therefore, there is an absolute constant $c_2 > 0$ such that

$$l \geqslant c_2/(2q) \Rightarrow \mathbb{P}[\mathscr{D}_r^m(s;\eta) \geqslant \tfrac{1}{2}\beta_1 q l \mid \mathscr{D}_R^m(0;\eta) \geqslant l] \geqslant 1/2 \tag{2.45}$$

Denote by F the distribution of the random variable $\mathscr{D}_R^m(0;\eta)$ and assume that $\mathbb{E}[\mathscr{D}_R^m(0;\eta)] = L_1 \geqslant c_2/q$. In the following calculation, (2.45) is used to validate the last but one inequality:

$$\begin{aligned}
\mathbb{E}[\mathscr{D}_r^m(s;\eta)] &\geqslant \int_{l \geqslant c_2/(2q)} \mathbb{E}[\mathscr{D}_r^m(s;\eta) \mid \mathscr{D}_R^m(0;\eta) = l]\, F(dl) \\
&\geqslant \int_{l \geqslant c_2/(2q)} \frac{\beta_1 q l}{2} \mathbb{P}\left[\mathscr{D}_r^m(s;\eta) \geqslant \frac{\beta_1 q l}{2} \middle| \mathscr{D}_R^m(0;\eta) = l\right] F(dl) \\
&\geqslant \frac{\beta_1 q}{4}\left[L_1 - \frac{c_2}{2q}\mathbb{P}\left[\mathscr{D}_R^m(0;\eta) < \frac{c_2}{2q}\right]\right] \geqslant \frac{\beta_1 q L_1}{8} \quad \blacksquare
\end{aligned}$$

Remark 2.11. Assume η_t, $t \geqslant 0$, stads for the process in which the particles which are present in η_0 execute independent random walks. The proof of Lemma 2.10 applies for this process also. Compared to the original proof (Lemma 4.3 through Corollary 1 in BL) designed for this process, our reasoning is based on the same ideas, though we avoid the use of the generating function and the consequent large-deviation estimate.

Lemma 2.12 (Counterpart of Lemma 4.5, BL). Suppose that ξ_0 is translation invariant with $\mathbb{E}[\mathscr{D}_R^m(0)] \geqslant L_1$, where $L_1 \geqslant c_2/q$ for appropriate $c_2 > 0$. Then, for sufficiently large t, either

$$\mathbb{E}[\mathscr{D}_{r,j}^m(s;\xi)] \geqslant \beta_1 q L_1/16$$

for all $s \in [R^2/2, R^2]$ and all $j \in J$, or

$$\mathbb{E}[\mathscr{D}_R^T(0)] - \mathbb{E}[\mathscr{D}_R^T(R^2)] \geqslant \beta_1 L_1/16$$

Proof. This lemma is proven absolutely analogously to its counterpart. The proof stems from the estimate (2.41) provided by Lemma 2.10 and the fact that $\xi_0 = \eta_0 \Rightarrow \xi_t \subseteq \eta_t$ $\forall t \geqslant 0$, as follows from the constructions (2.1) and (2.2). The values of the absolute constants c_2 and β_1 above have been specified in the course of the proof of Lemma 2.10. $\blacksquare$

Lemma 2.13 (Counterpart of Lemma 4.6, BL). Let $R \equiv R_t$ be given by (iv) of Section 2.4. Suppose that ξ_0 is translation invariant with $\mathbb{E}[\mathscr{D}_R^m(0)] \geqslant L_1$, where $L_1 \geqslant c_2/q$ for appropriate c_2. Then for appropriate β_3 (not depending on δ_1, δ_2) and large enough t,

$$\mathbb{E}[\mathscr{D}_R^T(0)] - \mathbb{E}[\mathscr{D}_R^T(R^2)] \geqslant \beta_3 L_1$$

Proof. The proof of this lemma is based on Lemmas 2.9 and 2.12 and the particular choice of R_t and r_t given in Definition 2.4. The proof is identical to that of Lemma 4.6, BL. β_3 above depends exclusively on the absolute constants introduced until now. ▮

The Proof of Theorem 1.1. We emphasize that the following proof does not contain new ideas compared to the original argument used in BL to obtain an upper bound for the density of the two-particle annihilating random walks. However, we include this proof since it makes the paper self-contained and easier to understand.

Fix $0<\varepsilon<1/8$. Let t_3 be large enough so that Lemma 2.13 holds and larger than $t_2(\varepsilon)$ for the dimensions $d\leqslant 4$ and $t_2(1/8)$ in the dimensions $d>4$, where t_2 is from Lemma 2.6. Assume that

$$\text{for some} \quad t>t_3 \quad \text{it holds that} \quad \rho(t)>g(t) \tag{2.46}$$

where

$$g(t)=\begin{cases} Ct^{2\varepsilon}t^{-d/4} & \text{for} \quad d\leqslant 4 \\ Ct^{-1} & \text{for} \quad d>4 \end{cases} \tag{2.47}$$

and $C=C(d)$ is an absolute constant whose value will be specified below in the proof.

Since the density is nonincreasing in time, the above assumption yields that for all $s\leqslant t$,

$$\mathbb{E}[\mathscr{D}_R^T(s)]\geqslant\mathbb{E}[\mathscr{D}_R^T(t)]=2R_t^d\rho(t)>\begin{cases} C_7t^{2\varepsilon}t^{d/4} & \text{for} \quad d\leqslant 4 \\ C_7t^{d/2-1} & \text{for} \quad d>4 \end{cases} \tag{2.48}$$

Due to Lemma 2.6,

$$\mathbb{E}[|\mathscr{D}_R(s)|]\leqslant C_6R_t^{d/2}t^{\varepsilon}=C_6(\delta_1 t)^{d/4}\,t^{\varepsilon} \qquad \text{for all} \quad s\in[t/2,t]$$

Comparing this to (2.48), we conclude that

$$\mathbb{E}[|\mathscr{D}_R(s)|]\leqslant(\mathbb{E}[\mathscr{D}_R^T(s)])/3, \qquad \forall s\in[t/2,t] \tag{2.49}$$

Observe that from the definitions of $\mathscr{D}_R^T(\cdot)$, $\mathscr{D}_R^m(\cdot)$, and $|\mathscr{D}_R(\cdot)|$, it follows that

$$\mathbb{E}[\mathscr{D}_R^T(s)]=2\mathbb{E}[\mathscr{D}_R^m(s)]+\mathbb{E}[|\mathscr{D}_R(s)|] \qquad \text{for all} \quad s\geqslant 0 \tag{2.50}$$

From (2.49) and (2.50) we derive that

$$\mathbb{E}[\mathscr{D}_R^T(s)]<3\mathbb{E}[\mathscr{D}_R^m(s)] \qquad \text{for all} \quad s\in[t/2;t] \tag{2.51}$$

Choose δ_1 in (iv) of Section 2.4 such that it would possible to insert $24/\beta_3$ intervals of the length R_t^2 in the interval $[t/2;t]$. Let $t/2=s_0, s_1,..., s_{k-1}$,

$s_k = t$ be the endpoints of these intervals. Using (2.48), it is easy to see that the assumption (2.46) yields $\mathbb{E}[\mathscr{D}_R^T(t/2)] > 24c_2 q^{-1} = 24c_2 R_t^d r_t^{-d}$ when the constant C is chosen in an appropriate way, depending exclusively on the absolute constants introduced until now. Because of the last inequality, we now can state that

$$\text{there is } n \text{ such that} \quad \mathbb{E}[\mathscr{D}_R^m(s_n)] \leqslant \mathbb{E}[\mathscr{D}_R^T(t/2)]/24 \tag{2.52}$$

In fact, assuming (2.52) is wrong, we then would be able to conclude, based on Lemma 2.13, that

$$\mathbb{E}[\mathscr{D}_R^T(t/2)] - \mathbb{E}[\mathscr{D}_R^T(t)] = \sum_{i=0}^{k-1} (\mathbb{E}[\mathscr{D}_R^T(s_i)] - \mathbb{E}[\mathscr{D}_R^T(s_{i+1})]) > \mathbb{E}[\mathscr{D}_R^T(t/2)]$$

which is impossible. Using (2.51), (2.52), and the fact that $\mathbb{E}[\mathscr{D}_R^T(\cdot)]$ is nonincreasing in time, we conclude that

$$\mathbb{E}[\mathscr{D}_R^T(t)] \leqslant \mathbb{E}[\mathscr{D}_R^T(s_n)] < 3\mathbb{E}[\mathscr{D}_R^m(s_n)] \leqslant \mathbb{E}[\mathscr{D}_R^T(t/2)]/8$$

In other words, $\rho(t)/\rho(t/2) < 1/8$. Observe now that when ε is sufficiently small, $g(t)/g(t/2) > 1/4$ holds for all $t > 0$. Thus, the conclusion is that the assumption (2.46) yields the following relationship:

$$\frac{\rho(t/2)}{g(t/2)} > 2\frac{\rho(t)}{g(t)}$$

A consequence from this conclusion is that, for any $t > t_8$, there is a point in the sequence $t, 2t, 4t,...$ at which the ratio $\rho(\cdot)/g(\cdot)$ drops beneath 1. Assume t_0 is such that $\rho(t_0)/g(t_0) < 1$. Now, if $t = 2t_0$ satisfies the assumption (2.46), then, applying the same argument, one has

$$\frac{\rho(t_0)}{g(t_0)} > 2\frac{\rho(2t_0)}{g(2t_0)} > 2$$

which is impossible. Therefore, $\rho(2^n t_0)/g(2^n t_0) < 1$ for all $n \in \mathbb{N}$. But for each $s \in [2^n t_0; 2^{n+1} t_0]$, we have that $\rho(s) \leqslant \rho(2^n t_0)$, while $4g(s) > g(2^n t_0)$. Thus, $\rho(t)/g(t) < 4$ for all $t \geqslant t_0$. Recalling the definition of $g(t)$ gives the assertion of the theorem. ▮

Remark 2.14. In the proof of Theorem 1.1, we used the fact that the assumption (2.46) implied $\mathbb{E}[|\mathscr{D}_R(s)|] \leqslant (\mathbb{E}[\mathscr{D}_R^T(s)])/3$ for $s \in [t/2, t]$ when t is sufficiently large. This implication will be true also if we choose

$$g_\varepsilon(t) = t^{2\varepsilon} t^{-d/4} \qquad \text{for} \quad d > 4 \tag{2.53}$$

This fact may give an impression that the true upper bound should behave as $\sim(t^{-d/4})$ also when $d > 4$. This impression is incorrect, however. The reason is that for the needs of the proof, we must guarantee that $\mathbb{E}[\mathscr{D}_R^T(t/2)] > 24c_2/q$ for all sufficiently large t. Recall that $q = (R_t/r_t)^d$. The last inequality is therefore valid provided $\rho(t/2) > 12c_2 r_t^{-d} = c_4 t^{-1}$; thus, under the choice (2.53) and the assumption (2.46), the whole proof breaks down. Also observe that we cannot take r_t larger than $O(t^{1/d})$ because for the needs of the proof of Lemma 2.13 we have to maintain $R_t^2/r_t^d = \text{const}$ [see (4.31) in the proof of the counterpart of this lemma in BL].

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